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ADDENDUM

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Abstract. We prove by induction that, with certain fixed boundary conditions, elementary cellular automaton rule 3 has a period-two limit cycle to which all states are attracted, regardless of lattice size.

In a recent letter [1], we proposed a classification of cellular automata (CA) in close analogy with discrete dynamical systems. This classification originated from an extensive numerical study, which left open the question of what happens in the large-lattice limit.

Since the submission of [1], we have made considerable progress in understanding rigorously CA rules with a lattice size-independent global attractor. In this addendum we present an inductive proof of the existence of such an attractor for a particular rule, and comment on other rules.

We now proceed to define elementary cellular automaton rule 3. In the remainder of this paper s will stand for the state of a lattice site, which can be 0 or 1, and S for the state of a collection of sites. The overbar stands for logical complement, and the dot for logical and.

Definition 1. The evolution of rule 3 is given by

$$s_i^{t+1} = \bar{s}_{i-1}^t \cdot \bar{s}_i^t \ .$$

In other words, s_i becomes 1 at $t+1$ if it and its left neighbour are zero at time t, and becomes 0 otherwise.

Property 1. The state of a site at $t+1$ is independent of the state of its right neighbour at time t.

The theorem we wish to prove is the following.

Theorem 1. Under fixed boundary conditions of zero to the left and right of the lattice, the rule 3 automaton has a period-two limit cycle which attracts all initial states, independently of lattice size.

In the remainder of this proof fixed boundaries of zero to the left and right of the lattice will be implicitly assumed. We also use S', s' to denote lattice and site states at time $t+1$.

Proposition 1. The states 0^L and 1^L map to each other for all L.

Proof. Since $s'(000) = 1$, then $S'(0^L) = 1^L$. Since $s'(010, 011, 111, 110) = 0$, then $S'(1^L) = 0^L$. This shows that there is a period-two limit cycle for all L. □

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Proposition 2. The state $0^{L-1}1$ maps to 0^L after two time steps for all L.

Proof. $S'(0^{L-1}1) = 1^{L-1}0$, from definition 1. $S'(1^{L-1}0) = 0^L$, from definition 1. □

We now prove that all states eventually map to 0^L for $L = 1$ and 2, and therefore are attracted to the period-two cycle shown in proposition 1.

Lemma 1. Both one-site states eventually map to 0.

Proof. Follows from proposition 1, with $L = 1$. □

Lemma 2. All four two-site states eventually map to 00.

Proof. $S'(01) = 10$, $S'(10, 11) = 00$, $S'(00) = 11$, from definition 1. □

Therefore, all states are attracted to the limit cycle 00–11–00....

Proof of Theorem 1. All states of length $L+1$ are formed by adding a 0 or 1 to states of length L. We now show that, if a state S of length L is mapped to 0^L, the states $S0$ and $S1$ are mapped to 0^{L+1}, and by proposition 1, to the cycle $0^{L+1} - 1^{L+1} - 0^{L+1} \ldots$.

Suppose that S eventually maps to 0^L after T time steps. Then, by property 1, $S0$ and $S1$ map to either 0^L0 or 0^L1. In the first case, we have the desired result. Furthermore we can say that $S0$ or $S1$ map to 0^{L+1} after T time steps. In the second case, by using proposition 2, we see that $S0$ or $S1$ map to 0^{L+1} after $T+2$ time steps.

Since it has been shown (lemmas 1 and 2) that for $L = 1$ and 2 all states eventually map to 0^L, the above proves that all s^L states eventually map to 0^L for $L > 2$ as well. We therefore know by proposition 1 that all states evolve into a period-two cycle. □

The state diagram of this system is peculiar, since all states other than 0^L, 1^L are injected into the period-two cycle through one state, 0^L. In this way, the automaton differs from the discretized iterated maps, in which any cycle state can be reached first.

From lemma 2, which shows that the state 01 takes four time steps to reach the limit cycle, and the main proof, we can deduce the following property of the transients.

Corollary 1. For a lattice of length L the longest transient is $2(L-1)$.

This transient corresponds to the initial state 01^{L-1} which evolves to 0^L in $2(L-1)$ time steps. The odd-time states correspond to successive shifts to the right of an isolated 1 site. From property 1 we see that the proof also applies if a fixed boundary condition of 1 is applied to the right of the lattice. With a fixed left boundary of 1, there is also a period-two global cycle, $0^L - 01^{L-1} - 0^L \ldots$ lemma 1 is then no longer true, and the theorem can only be proved strictly for $L > 1$.

We have proved that with certain fixed boundary conditions, elementary cellular automaton rule 3 has a period-two global attractor independently of lattice size. Such behaviour corresponds to a periodic attractor in discretized versions of iterated maps. With this work, we have established a novel feature of cellular automata which brings them closer to dissipative dynamical systems.

In [1] a total of 14 rules were found numerically to exhibit lattice size-independent global attractors of lengths 2, 3, 4 or 6. The present method of proof works for most of them. In most cases, the limit cycle consists of left or right shifting of repeated (001) or (0011) blocks.

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References

[1] Binder P-M 1991 *J. Phys. A: Math. Gen.* **24** L31